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Multiple spatial and temporal scales method for numerical simulation of non-classical heat conduction problems: one dimensional case

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Abstract

A multiple spatial and temporal scales method is studied to simulate numerically the phenomenon of non-Fourier heat conduction in periodic heterogeneous materials. The model developed is based on the higher-order homogenization theory with multiple spatial and temporal scales in one dimensional case. The amplified spatial scale and the reduced temporal scale are introduced respectively to account for the fluctuations of non-Fourier heat conduction due to material heterogeneity and non-local effect of the homogenized solution. By separating the governing equations into various scales, the different orders of homogenized non-Fourier heat conduction equations are obtained. The reduced time dependence is thus eliminated and the fourth-order governing differential equations are derived. To avoid the necessity of C^1 continuous finite element implementation, a C^0 continuous mixed finite element approximation scheme is put forward. Numerical results are shown to demonstrate the efficiency and validity of the proposed method.

Keywords: Non-Fourier heat conduction; Multiple scale method; Homogenization; Non-local model

1. Introduction

The classical Fourier law is well known and has been used successfully for analysis of steady heat conduction process under long time heating and unsteady process with quick propagation speed of the thermal wave. However, Fourier law breaks down in situations involving very short times, high heat fluxes, and at

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very cryogenic temperatures (Baumeister and Hamill, 1969). The anomaly of this classical theory is from the assumption that the heat flux vector and the temperature gradient across a material volume occur at the same instant of time. Such an immediate response results in an infinite speed of heat propagation.

The heat sources such as laser and microwave with very high frequency and extremely short duration have been used widely in modern technology in past years. This leads to the increase of research interest of non-Fourier law. The mathematical description of non-Fourier heat conduction law, which represents the time lag of heat waves, is a hyperbolic type differential equation. As has been pointed out by many researchers, this non-classical heat conduction law has its great value in many practical applications, such as laser penetration and welding, explosive bonding, electrical discharge machining, heating and cooling of microelectronic elements involving a duration time of nanoseconds or even picoseconds in which the energy is absorbed within a distance of microns from the surface. In order to associate a finite heat propagation speed, Cattaneo (1958) and Vernotte (1961) modified Fourier law by including a relaxation model. Non-Fourier heat conduction in solids with different shapes and boundary conditions has been studied extensively. Frankel et al. (1987), using flux formulation of hyperbolic heat conduction equation, gave an analytical solution for a finite slab under boundary condition of rectangular heat pulse. Ozisik and Tzou (1994) analyzed the special features in thermal wave propagation, and the thermal wave model in relation to the microscopic two-step model. Kaminski (1990) determined experimentally the values of a relaxation time for non-homogeneous inner structure materials. Tzou (1995) presented a universal constitutive equation between the heat flux vector and the temperature gradient. Jiaung et al. (2003) studied the effect of the phase lag of temperature gradient. On the other hand, the stochastic finite element method was successfully applied in displacement-based finite element method in transient heat transfer for heterogeneous media, which is based on the second order perturbation second central probabilistic moment method (Hien and Kleiber, 1997; Kamiński and Hien, 1999a,b).

It has been found that the multi-scale asymptotic homogenization approach is wide acceptance for the study of heterogeneous structures due to its systematic mathematical approach and ability to account for multi-scale features (Bakhvalov and Panasenko, 1989; Benssousan et al., 1978; Chung et al., 2004; Sanchez-Palencia, 1980). The mathematical homogenization method was used as an alternative approach to compute effective constitutive parameters of complex materials with a periodic structure in Hassani and Hinton (1998). To capture the effects of microstructural changes on the overall response of a composite made of bodies in elastic and elastic–plastic contact, numerical homogenized constitutive law is then defined in Zhang et al. (1999) and Zhang and Schrefler (2000) for the global behavior of the heterogeneous materials. For the composite with detailed information on the microgeometry, Kamiński (2000) extended the effective modules method by using the finite element method or the boundary element method in numerical implementations, which enable direct computations of the effective characteristics. Gambin and Kroner (1989), and Boutin (1996) have studied the role of higher-order terms in the asymptotic expansion in statics. Boutin and Auriault (1993) demonstrated the terms of a higher-order successively introduced effects of dispersion and attenuation in elastokinetics. A single-frequency time-dependence is assumed prior to the homogenization process (Kevorkian and Bosley, 1998). Chen and Fish (2001) and Fish and Chen (2001a,b) investigated the problem of secularity introduced by the higher order multiple spatial–temporal scale approximation of the initial boundary value problem in periodic heterogeneous media. Fish et al. (2002) developed a non-local approach independent of the slow time scale considered the problem of secularity.

Recently, considerable interest has been generated toward transient heat transfer by the multi-scale asymptotic homogenization method and its potential applications in engineering and technology. Boutin (1995) studied the heat propagation in media with a periodic microstructure. It is shown that the higher terms introduce successive gradients of temperature and tensors, characteristic of the microstucture, which result in non-local effects. A systematic way of obtaining the effective viscoelastic moduli in time and frequency domain is presented for periodic microstructures in Yi et al. (1998), the effective modulus is formulated using the asymptotic homogenization method. Yu and Fish (2002) developed a systematic approach to analyzed multiple physical processes interacting at multiple spatial and temporal scales, which be applied to the coupled thermo-viscoelastic composites with microscopically periodic mechanical and thermal properties. Kamiński (2003) applied the mathematical model to the homogenization of transient heat transfer problems in some composite materials, where the finite element method computation is based on the effective modules method introduced for periodic composites.

The modeling approach in this paper differs somewhat from those proposed in previous studies since in the proposed present analysis, non-Fourier heat conduction law is adopted to describe the heat conduction process. To resolve the dispersion effect, following the work contributed by Fish et al. (2002), a computational model based on the higher-order homogenization with multiple spatial and temporal scales is developed. The wave behaviors of non-Fourier heat conduction in periodic heterogeneous materials subjected to extreme conditions are investigated. By introducing amplified spatial scale and reduced temporal scale in the multiscale analysis model, different orders of homogenized equations are derived from the mathematical homogenization theory. With incorporating different orders of homogenized heat conduction equations and eliminating a reduced temporal scale dependence, the high-order homogenized heat conduction equation at the macro scale is obtained. To avoid the requirement of C^1 continuous finite element in numerical implementation, the C^0 continuous mixed finite element approximation is developed for a solution of the resulting non-local non-Fourier heat conduction equations. Finally, numerical results are given to illustrate the efficiency and validity of the method proposed.

2. Governing equations of heat conduction with non-Fourier law

In the analysis of heat conduction, two kinds of constitutive equation for heat conduction can be adopted. The first, and the most widely employed one is the classical Fourier law

$$q(x,t) = k(x)\phi(x,t)_{;x} \tag{1}$$

With Eq. (1) and the local energy balance equation

$$q(x,t)_{;x} = \rho c\dot{\phi}(x,t) \tag{2}$$

the following classical parabolic heat conduction equation can be obtained

$$\rho(x)c(x)\dot{\phi}(x,t) - \{k(x)\phi(x,t)_{;x}\}_{;x} = 0 \tag{3}$$

where $q(x,t)$ is the heat flux density, $k(x,t)$ is a positive scalar representing thermal conductivity of the material, $\phi(x,t)$ is the local equilibrium temperature, $\rho(x,t)$ is the mass density and $c(x,t)$ is the specific heat capacity. As it is well known, Eq. (3) yields a parabolic differential equation for the temperature field. $(\ )_{;x}$ and $(\dot{\cdot})$ denote the total derivatives with respect to space and time variable, respectively.

As pointed out by many researchers, the classical Fourier heat conduction theory becomes inaccurate due to neglecting the effect of a finite speed of propagation when heat sources such as lasers and microwaves with extremely short duration or very high frequency, are investigated. Under this circumstance, non-Fourier heat conduction model becomes more reliable in describing the diffusion process and predicting the temperature distribution. A non-Fourier heat flux model can be expressed in the following form

$$q(x,t) = k(x)\phi(x,t)_{;x} + \tau(x)\dot{q}(x,t) \tag{4}$$

With the help of Eqs. (2) and (4), a hyperbolic differential governing equation of non-Fourier heat conduction can be derived

$$\lambda(x)\{\tau(x)\ddot{\phi}(x,t) + \dot{\phi}(x,t)\} - \{k(x)\phi_{;x}\}_{;x} = 0 \tag{5}$$

where $\tau(x)$ is the relaxation time, $\lambda(x)$ is the specific heat of unit volume.

3. Asymptotic analysis method with multiple spatial and temporal scales

In the present paper, one dimensional non-classical heat conduction process in periodic heterogeneous material will be investigated. Following the work presented by Fish et al. (2002), it is assumed that the characteristic size of the macroscopic domain L is sufficiently larger than the characteristic size l of the material heterogeneity. Two spatial variables: a macro- or whole spatial scale x and a micro- or local amplified spatial scale y, which are related by (see Fig. 1)

$$y = x/\varepsilon \tag{6}$$

where $0 < \varepsilon \ll 1$ denotes the amplified spatial variation of material properties.

In addition to the spatial variables with different scales, the two multiple scale time variables are also introduced: one is a general temporal scale $t_0 = t$ and the other is a reduced temporal scale $t_1 = \varepsilon^2 t$.

Since the transient temperature field ϕ depends on x, y, t_0 and t_1, a multiple-scale asymptotic expansion is employed to approximate the transient temperature field ϕ

$$\phi(x,y,t) = \phi_0(x,y,t_0,t_1) + \varepsilon\phi_1(x,y,t_0,t_1) + \varepsilon^2\phi_2(x,y,t_0,t_1) + \cdots \tag{7}$$

Let us consider a two-component bar with the periodic microstructures as shown in Fig. 1, the input heat supply $\phi(t)$ on the right end of the bar is the only heat source. The adiabatic boundary conditions are imposed on the bar, so that no heat transfer occurs between the bar and the ambience. l is the unit cell dimension on x spatial scale and $\widehat{\Omega}$ are the length of the unit cell on y spatial scale, where $\widehat{\Omega} = l/\varepsilon$. In the following, we assume that the spatial gradient term of the heat flux density has local periodicity.

The hyperbolic equation governing the transient non-Fourier heat conduction is given by

$$\lambda(x/\varepsilon)\{\tau(x/\varepsilon)\ddot{\phi} + \dot{\phi}\} - \{k(x/\varepsilon)\phi_{;x}\}_{;x} = 0 \quad \text{on } \Omega \tag{8}$$

The macro-domain boundary conditions are

$$\phi(0,t) = 0 \tag{9}$$

$$\phi_{;x}(L,t) = \frac{Q(t)}{k(L)\cdot F} \tag{10}$$

The initial boundary conditions can be written as

$$\phi(x,0) = f(x) \tag{11}$$

$$\dot{\phi}(x,0) = g(x) \tag{12}$$

where F and $Q(t)$ are cross-sectional area of the bar and external heat source, respectively; Ω is the entire macro domain (λ (x/ε), $\tau(x/\varepsilon)$ and $k(x/\varepsilon)$ have local periodicities on microscopic constitution).

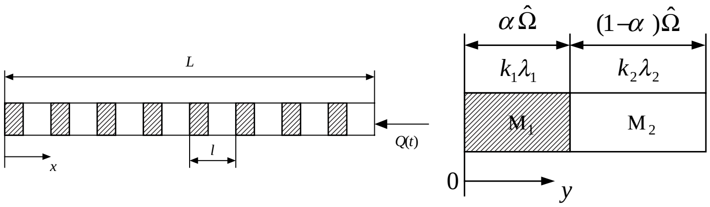

Fig. 1. One-dimensional bar with periodic microstructure and the associated unit cell.

The following results in Hassani and Hinton (1998) will be used in the following derivations:

Fact (1). The derivative of a periodic function is also a periodic function with the same period.

Fact (2). The integral of the derivative of a periodic function over the period is zero. (These facts can easily be verified by the definition of derivative and periodicity).

Fact (3). If $\Phi = \Phi(x, y(x))$, then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\cdot\frac{\partial y}{\partial x} \tag{13}$$

if $y = x/\varepsilon$, then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\partial\Phi}{\partial x} + \frac{1}{\varepsilon}\cdot\frac{\partial\Phi}{\partial y} \tag{14}$$

The averaging operator is defined as

$$\langle\cdot\rangle = |Y|^{-1}\int_Y \mathrm{d}Y \tag{15}$$

Using the chain rule, the spatial and temporal derivatives can be expressed as

$$\phi_{;x} = \phi_{,x} + \varepsilon^{-1}\phi_{,y}, \quad \dot{\phi} = \phi_{,t_0} + \varepsilon^2\phi_{,t_1}, \quad \ddot{\phi} = \phi_{,t_0t_0} + 2\varepsilon^2\phi_{,t_1t_0} + \varepsilon^4\phi_{,t_1t_1} \tag{16}$$

Then the spatial gradient term of the heat flux density takes the following form

$$q' = k(\phi_{,x} + \varepsilon^{-1}\phi_{,y}) \tag{17}$$

Substituting Eq. (16) into Eq. (8), yields

$$\lambda\cdot\{(\tau\phi_{,t_0t_0} + \phi_{,t_0}) + \varepsilon^2(2\tau\phi_{,t_0t_1} + \phi_{,t_1}) + \varepsilon^4\tau\phi_{,t_1t_1}\} = q'_{,x} + \varepsilon^{-1}q'_{,y} \tag{18}$$

Substituting the asymptotic expansion of $\phi(x, y, t)$ into Eq. (17), we obtain the asymptotic expansion of q'.

$$\begin{aligned} q' &= k\{(\phi_{0,x} + \varepsilon^{-1}\phi_{0,y}) + (\varepsilon\phi_{1,x} + \phi_{1,y}) + (\varepsilon^2\phi_{2,x} + \varepsilon\phi_{2,y}) + \cdots\} \\ &= k\{\varepsilon^{-1}\phi_{0,y} + (\phi_{0,x} + \phi_{1,y}) + \varepsilon(\phi_{1,x} + \phi_{2,y}) + \cdots\} \\ &= \varepsilon^{-1}q'_{-1} + q'_0 + \varepsilon q'_1 + \varepsilon^2 q'_2 + \cdots \end{aligned} \tag{19}$$

where

$$q'_{-1} = k\phi_{0,y}, \quad q'_s = k(\phi_{s,x} + \phi_{s+1,y}) \quad s = 0, 1, 2, \ldots \tag{20}$$

Substituting the asymptotic expansions of $\phi(x, y, t)$ and q' into Eq. (18), LHS of Eq. (18) becomes

$$\begin{aligned} &\lambda\cdot\{(\tau\phi_{0,t_0t_0} + \phi_{0,t_0}) + \varepsilon(\tau\phi_{1,t_0t_0} + \phi_{1,t_0}) + \varepsilon^2(\tau\phi_{2,t_0t_0} + \phi_{2,t_0}) + \cdots + \varepsilon^2(2\tau\phi_{0,t_0t_1} + \phi_{0,t_1}) \\ &\quad + \varepsilon^3(2\tau\phi_{1,t_0t_1} + \phi_{1,t_1}) + \varepsilon^4(2\tau\phi_{2,t_0t_1} + \phi_{2,t_1}) + \cdots + \varepsilon^4\tau\phi_{0,t_1t_1} + \varepsilon^5\tau\phi_{1,t_1t_1} + \varepsilon^6\tau\phi_{2,t_1t_1} + \cdots\} \end{aligned} \tag{21}$$

RHS of Eq. (18) becomes

$$\begin{aligned} &\varepsilon^{-1}q'_{-1,x} + q'_{0,x} + \varepsilon q'_{1,x} + \cdots + \varepsilon^{-2}q'_{-1,y} + \varepsilon^{-1}q'_{0,y} + q'_{1,y} + \cdots \\ &\quad = \varepsilon^{-2}q'_{-1,y} + \varepsilon^{-1}(q'_{0,y} + q'_{-1,x}) + (q'_{1,y} + q'_{0,x}) + \cdots \end{aligned} \tag{22}$$

Comparing Eqs. (21) and (22), since the corresponding coefficients of the same order of ε^n are equivalent, the following equations of heat conduction for various orders of ε are obtained

$$\mathrm{O}(\varepsilon^{-2}) : q'_{-1,y} = 0 \tag{23}$$

$$\mathrm{O}(\varepsilon^{-1}) : q'_{0,y} + q'_{-1,x} = 0 \tag{24}$$

$$\mathrm{O}(\varepsilon^{0}) : \lambda(\tau\phi_{0,t_0t_0} + \phi_{0,t_0}) = q'_{1,y} + q'_{0,x} \tag{25}$$

$$\mathrm{O}(\varepsilon^{1}) : \lambda(\tau\phi_{1,t_0t_0} + \phi_{1,t_0}) = q'_{2,y} + q'_{1,x} \tag{26}$$

$$\mathrm{O}(\varepsilon^{2}) : \lambda(\tau\phi_{2,t_0t_0} + \phi_{2,t_0} + 2\tau\phi_{0,t_0t_1} + \phi_{0,t_1}) = q'_{3,y} + q'_{2,x} \tag{27}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

4. Resolution of problems at various orders of heat conduction equations

4.1. $O(\boldsymbol{\varepsilon}^{-2})$ homogenization problem

Consider the $\mathrm{O}(\varepsilon^{-2})$ heat conduction equation (23), i.e. $q'_{-1,y} = 0$, multiplying it by ϕ_0, then integrating it over the unit cell domain, and finally performing integration by parts yields

$$\int_Y \phi_0 q'_{-1,y}\,\mathrm{d}Y = \int_{\partial Y} \phi_0 q'_{-1} n\,\mathrm{d}s - \int_Y k(\phi_{0,y})^2\,\mathrm{d}Y = 0 \tag{28}$$

The first term in Eq. (28) vanishes because of periodicity of the boundary conditions in the unit cell. Since k is a positive scalar, we have

$$\phi_{0,y} = 0 \Rightarrow \phi_0 = \Phi_0(x, t_0, t_1) \tag{29}$$

and

$$q'_{-1} = k\phi_{0,y} = 0 \tag{30}$$

4.2. $O(\varepsilon^{-1})$ homogenization problem

Consider the $\mathrm{O}(\varepsilon^{-1})$ heat conduction equation (24), i.e. $q'_{0,y} + q'_{-1,x} = 0$, since Eq. (30) $q'_{-1} = 0$ results in $q'_{-1,x} = 0$, substituting Eqs. (20) and (29) into Eq. (24) yields

$$q'_{0,y} + q'_{-1,x} = q'_{0,y} = \{k(\phi_{1,y} + \Phi_{0,x})\}_{,y} = 0 \tag{31}$$

Taking the linear relationship between ϕ_1 and $\Phi_{0,x}$ into consideration, we can write the general form of ϕ_1 as follows

$$\phi_1(x, y, t_0, t_1) = \Phi_1(x, t_0, t_1) + A(y)\Phi_{0,x} \tag{32}$$

Substituting Eq. (32) into Eqs. (31) and (20) yields

$$\{k(1 + A_{,y})\}_{,y} = 0 \tag{33}$$

$$q'_0 = \Phi_{0,x} k(1 + A_{,y}) \tag{34}$$

Consider the structure of unit cell shown in Fig. 1, the cell domain consists of subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ occupied by microconstituents 1 and 2, respectively, such that

$$\Omega^{(1)} = \{y|0 \leqslant y < \alpha\widehat{\Omega}\}, \quad \Omega^{(2)} = \{y|\alpha\widehat{\Omega} \leqslant y \leqslant \widehat{\Omega}\}$$

where $0 \leqslant \alpha \leqslant 1$ is the volume fraction of material 1 in the unit cell. Eqs. (33) and (34) can be rewritten over a unit cell domain

$$k_j(1 + A_{j,y}) = a_j, \quad q'_{0(j)} = \Phi_{0,x}k_j(1 + A_{j,y}) \quad (j = 1, 2) \tag{35}$$

where a_j is constants. $A_j(y)$ can be solved from the following conditions
Periodicity conditions

$$\phi_1(y = 0) = \phi_1(y = \widehat{\Omega}) \tag{36}$$

$$q'_0(y = 0) = q'_0(y = \widehat{\Omega}) \tag{37}$$

Continuity conditions

$$[\phi_1(y = \alpha\widehat{\Omega})] = 0 \tag{38}$$

$$[q'_0(y = \alpha\widehat{\Omega})] = 0 \tag{39}$$

where [ ] is the jump operator.

Taking normalization, we have

$$\langle\phi_1(x, y, t_0, t_1)\rangle = \Phi_1(x, t_0, t_1) \Rightarrow \langle A(y)\rangle = 0 \tag{40}$$

and $A_j(y)$ can be uniquely determined

$$A_1(y) = \frac{(1-\alpha)(k_2 - k_1)}{(1-\alpha)k_1 + \alpha k_2}\left(y - \frac{\alpha\widehat{\Omega}}{2}\right) \tag{41}$$

$$A_2(y) = \frac{\alpha(k_1 - k_2)}{(1-\alpha)k_1 + \alpha k_2}\left(y - \frac{(1+\alpha)\widehat{\Omega}}{2}\right) \tag{42}$$

$$k_n = \langle k(1 + A_{,y})\rangle = \frac{k_1 k_2}{(1-\alpha)k_1 + \alpha k_2} \tag{43}$$

where k_n is called the zero-order homogenized thermal conductivity of the composite and has the same form as that for the classical homogenized model.

4.3. $O(\varepsilon^0)$ homogenization problem

Consider the $O(\varepsilon^0)$ heat conduction equation (25), i.e. $\lambda \cdot (\tau\phi_{0,t_0t_0} + \phi_{0,t_0}) = q'_{1,y} + q'_{0,x}$, taking into account Fact (2) and the definition of averaging operator, we have

$$\langle q'_{1,y}\rangle = 0 \tag{44}$$

Taking the relaxation time $\tau = \tau_n$, applying the averaging operator to Eq. (25) and substituting Eqs. (29) and (44) into Eq. (25) result

$$\lambda_n(\tau\Phi_{0,t_0t_0} + \Phi_{0,t_0}) - \langle q'_{0,x}\rangle = 0 \tag{45}$$

in which

$$\lambda_n = \langle\lambda\rangle = \alpha\lambda_1 + (1-\alpha)\lambda_2, \quad \tau_n = \langle\tau\rangle = \alpha\tau_1 + (1-\alpha)\tau_2 \tag{46}$$

where λ_n is homogenized specific heat of unit volume, and τ_n is homogenized relaxation time.

Substituting Eq. (34) into Eq. (45) yields the classical homogenized macroscopic equation of heat conduction at O(ε^0)

$$\lambda_n(\tau\Phi_{0,t_0t_0}+\Phi_{0,t_0})-k_n\Phi_{0,xx}=0 \tag{47}$$

Substituting Eqs. (20), (34) and (29), (32) into Eq. (25) yields

$$\lambda(\tau\Phi_{0,t_0t_0}+\Phi_{0,t_0})=\{k(\phi_{2,y}+\Phi_{1,x}+A\Phi_{0,xx})\}_{,y}+k(1+A_{,y})\Phi_{0,xx} \tag{48}$$

Substituting Eq. (47) into Eq. (48) yields

$$\{k(\phi_{2,y}+\Phi_{1,x}+A\Phi_{0,xx})\}_{,y}=k_n\{\beta(y)-1\}\Phi_{0,xx} \tag{49}$$

where

$$\beta(y)=\lambda(y)/\lambda_n \tag{50}$$

Due to the linear relationship between ϕ_2 and $\Phi_{1,x}$, $\Phi_{0,xx}$, the general solution of ϕ_2 is decomposed as

$$\phi_2(x,y,t_0,t_1)=\Phi_2(x,t_0,t_1)+A(y)\Phi_{1,x}+B(y)\Phi_{0,xx} \tag{51}$$

Substituting Eq. (51) into Eqs. (49) and (20) yields

$$\{k(A+B_{,y})\}_{,y}=\{\beta(y)-1\}k_n \tag{52}$$

$$q'_1=k(1+A_{,y})\Phi_{1,x}+k(A+B_{,y})\Phi_{0,xx} \tag{53}$$

Following the same way and employing Eqs. (36)–(40), the solution of $B(y)$ can be derived as

$$\begin{aligned}B_1(y)=&\left\{\frac{k_n}{2k_1}\left(\frac{\lambda_1}{\lambda_n}-1\right)-\frac{(1-\alpha)(k_2-k_1)}{2((1-\alpha)k_1+\alpha k_2)}\right\}y^2+\left\{-\frac{\alpha\widehat{\Omega}k_n}{2k_1}\left(\frac{\lambda_1}{\lambda_n}-1\right)+\frac{\alpha(1-\alpha)\widehat{\Omega}(k_2-k_1)}{2((1-\alpha)k_1+\alpha k_2)}\right\}y\\&+\left\{-\frac{((1-\alpha^2)k_1-\alpha^2k_2)\alpha\widehat{\Omega}^2k_n}{12k_1k_2}\left(\frac{\lambda_1}{\lambda_n}-1\right)+\frac{\alpha(1-\alpha)(1-2\alpha)\widehat{\Omega}^2(k_2-k_1)}{12((1-\alpha)k_1+\alpha k_2)}\right\}\end{aligned} \tag{54}$$

$$\begin{aligned}B_2(y)=&\left\{\frac{k_n}{2k_2}\left(\frac{\lambda_2}{\lambda_n}-1\right)+\frac{\alpha(k_2-k_1)}{2((1-\alpha)k_1+\alpha k_2)}\right\}y^2\\&+\left\{-\frac{(1+\alpha)\widehat{\Omega}k_n}{2k_2}\left(\frac{\lambda_2}{\lambda_n}-1\right)+\frac{\alpha(1+\alpha)\widehat{\Omega}(k_2-k_1)}{2((1-\alpha)k_1+\alpha k_2)}\right\}y\\&+\left\{-\frac{((\alpha^3-3\alpha^2-3\alpha-1)k_1-(\alpha^3-\alpha^2)k_2)\widehat{\Omega}^2k_n}{12k_1k_2}\left(\frac{\lambda_2}{\lambda_n}-1\right)+\frac{\alpha(1+\alpha)(1+2\alpha)\widehat{\Omega}^2(k_2-k_1)}{12((1-\alpha)k_1+\alpha k_2)}\right\}\end{aligned} \tag{55}$$

According to Eqs. (54) and (55), we have

$$\langle\lambda A\rangle=0 \tag{56}$$

$$\langle k(A+B_{,y})\rangle=0 \tag{57}$$

4.4. O(ε^1) homogenization problem

Consider the O(ε^1) heat conduction equation (26), i.e. $\lambda(\tau\phi_{1,t_0t_0}+\phi_{1,t_0})=q'_{2,y}+q'_{1,x}$, taking into account Fact (2) and the definition of averaging operator, we have

$$\langle q'_{2,y}\rangle = 0 \tag{58}$$

Applying the averaging operator to Eq. (26) and then substituting Eqs. (32) and (58) into Eq. (26) yield

$$\lambda_n(\tau\Phi_{1,t_0t_0} + \Phi_{1,t_0}) + \langle\lambda A\rangle\{\tau(\Phi_{0,x})_{,t_0t_0} + (\Phi_{0,x})_{,t_0}\} = \langle q'_{1,x}\rangle \tag{59}$$

Substituting Eq. (53) into Eq. (59), yields

$$\lambda_n(\tau\Phi_{1,t_0t_0} + \Phi_{1,t_0}) + \langle\lambda A\rangle\{\tau(\Phi_{0,x})_{,t_0t_0} + (\Phi_{0,x})_{,t_0}\} = \langle k(1 + A_{,y})\rangle\Phi_{1,xx} + \langle k(A + B_{,y})\rangle\Phi_{0,xxx} \tag{60}$$

Substituting Eqs. (43), (56) and (57) into Eq. (60), yields the macroscopic equation of heat conduction at order of $O(\varepsilon^1)$

$$\lambda_n(\tau\Phi_{1,t_0t_0} + \Phi_{1,t_0}) - k_n\Phi_{1,xx} = 0 \tag{61}$$

Substituting Eqs. (20), (53) and (32) (51) into heat conduction equation (26) yields

$$\begin{aligned}\lambda\{(\tau\Phi_{1,t_0t_0} + \Phi_{1,t_0}) + A(\tau\Phi_{0,t_0t_0} + \Phi_{0,t_0})_{,x}\} &= \{k(\phi_{3,y} + \Phi_{2,x} + A\Phi_{1,xx} + B\Phi_{0,xxx})\}_{,y} \\ &\quad + k(1 + A_y)\Phi_{1,xx} + k(A + B_{,y})\Phi_{0,xxx}\end{aligned} \tag{62}$$

Substituting Eqs. (47) and (61) into Eq. (62) yields

$$\{k(\phi_{3,y} + \Phi_{2,x} + A\Phi_{1,xx} + B\Phi_{0,xxx})\}_{,y} = k_n(\beta - 1)\Phi_{1,xx} + \{k_n\beta A - k(A + B_{,y})\}\Phi_{0,xxx} \tag{63}$$

Due to the linear relationship between ϕ_3 and $\Phi_{2,x}$, $\Phi_{1,xx}$, $\Phi_{0,xxx}$, the general solution of ϕ_3 can be decomposed as

$$\phi_3(x, y, t_0, t_1) = \Phi_3(x, t_0, t_1) + A(y)\Phi_{2,x} + B(y)\Phi_{1,xx} + C(y)\Phi_{0,xxx} \tag{64}$$

Substituting Eq. (64) into Eqs. (63) and (20) yields

$$\{k(B + C_{,y})\}_{,y} = k_n\beta A - k(A + B_{,y}) \tag{65}$$

$$q'_2 = k_n\Phi_{2,x} + k(A + B_{,y})\Phi_{1,xx} + k(B + C_{,y})\Phi_{0,xxx} \tag{66}$$

Following the same way and employing Eqs. (36)–(40), the solution of $C(y)$ can be uniquely determined. After $C(y)$ is determined, we get

$$\langle\lambda B\rangle = \frac{(\alpha(1-\alpha))^2(\lambda_2 - \lambda_1)(k_1\lambda_1 - k_2\lambda_2)k_n\widehat{\Omega}^2}{12\lambda_n k_1 k_2} \tag{67}$$

$$\langle k(B + C_{,y})\rangle = -\frac{\alpha(1-\alpha)k_n\widehat{\Omega}^2}{12\lambda_n}\left\{\frac{(k_2 - k_1)(\alpha^2\lambda_1 - (1-\alpha)^2\lambda_2) + k_n\lambda_n}{(1-\alpha)k_1 + \alpha k_2} - \lambda_n\right\} \tag{68}$$

4.5. $O(\varepsilon^2)$ homogenization problem

Consider the $O(\varepsilon^2)$ heat conduction Eq. (27), i.e.

$$\lambda(\tau\phi_{2,t_0t_0} + \phi_{2,t_0} + 2\tau\phi_{0,t_0t_1} + \phi_{0,t_1}) = q'_{3,y} + q'_{2,x}$$

taking into account Fact (2) and the definition of averaging operator yields

$$\langle q'_{3,y}\rangle = 0 \tag{69}$$

Applying the averaging operator to Eq. (27) and then substituting Eqs. (29), (51) and (69) into Eq. (27) yield

$$\lambda_n\tau\Phi_{2,t_0t_0} + \tau\langle\lambda A\rangle(\Phi_{1,x})_{,t_0t_0} + \tau\langle\lambda B\rangle(\Phi_{0,xx})_{,t_0t_0} + \lambda_n\Phi_{2,t_0} + \langle\lambda A\rangle(\Phi_{1,x})_{,t_0} + \langle\lambda B\rangle(\Phi_{0,xx})_{,t_0} + 2\lambda_n\tau\Phi_{0,t_0t_1} + \lambda_n\Phi_{0,t_1} = \langle q_{2,x}\rangle \tag{70}$$

Substituting Eq. (66) into Eq. (70), yields

$$\lambda_n\tau\Phi_{2,t_0t_0} + \tau\langle\lambda A\rangle(\Phi_{1,x})_{,t_0t_0} + \tau\langle\lambda B\rangle(\Phi_{0,xx})_{,t_0t_0} + \lambda_n\Phi_{2,t_0} + \langle\lambda A\rangle(\Phi_{1,x})_{,t_0} + \langle\lambda B\rangle(\Phi_{0,xx})_{,t_0} + 2\lambda_n\tau\Phi_{0,t_0t_1} + \lambda_n\Phi_{0,t_1} = \langle k(1+A_{,y})\rangle\Phi_{2,xx} + \langle k(A+B_{,y})\rangle\Phi_{1,xxx} + \langle k(B+C_{,y})\rangle\Phi_{0,xxxx} \tag{71}$$

Substituting Eqs. (43), (56), (57), (67) and (68) into Eq. (71), yields the macroscopic equation of heat conduction at $O(\varepsilon^2)$

$$\lambda_n(\tau\Phi_{2,t_0t_0} + \Phi_{2,t_0}) - k_n\Phi_{2,xx} = k_d\Phi_{0,xxxx} - 2\lambda_n\tau\Phi_{0,t_0t_1} - \lambda_n\Phi_{0,t_1} \tag{72}$$

where

$$k_d = \langle k(B+C_{,y})\rangle - \langle\lambda B\rangle\lambda_n^{-1}k_n = \frac{(\alpha(1-\alpha))^2(k_1\lambda_1 - k_2\lambda_2)^2 k_n\widehat{\Omega}^2}{12\lambda_n^2((1-\alpha)k_1 + \alpha k_2)^2} \tag{73}$$

k_d is macroscopic characteristic due to structural heterogeneity.

Remark. For the sake of the further derivation of Eq. (45), we assume constant $\tau = \tau_n$, so the following formulation holds

$$\langle\lambda(y)\cdot\tau(y)\rangle = \langle\lambda(y)\rangle\cdot\langle\tau(y)\rangle \tag{74}$$

which will undoubtedly induce the doubt for the validity of the model developed. This will be checked by choosing different parameters of $\lambda(y)$ and $\tau(y)$, i.e. $\langle\lambda(y)\cdot\tau(y)\rangle \neq \langle\lambda(y)\rangle\cdot\langle\tau(y)\rangle$, in the real problems to be computed to discuss the validity of the high-order non-local model in which local fluctuations are introduced by material heterogeneity. This will be illuminated in the posterior numerical illustration (Section 7).

5. Non-local model

The macroscopic equations of heat conduction are stated in Eqs. (47), (61) and (72). The initial and boundary conditions for the above equations of heat conduction are prescribed as:

Initial conditions:

$$\begin{aligned} &\phi_0(x,0,0) = f(x), \quad \dot{\phi}_0(x,0,0) = g(x) \\ &\Phi_s(x,0,0) = 0, \quad \dot{\Phi}_s(x,0,0) = 0 \qquad s = 1,2 \end{aligned} \tag{75}$$

Boundary conditions:

$$\begin{aligned} &\phi_0(0,t_0,t_1) = 0, \quad \phi_{0;x}(L,t_0,t_1) = \frac{Q(t)}{k_n\cdot F} \\ &\Phi_s(0,t_0,t_1) = 0, \quad \Phi_{s;x}(L,t_0,t_1) = 0 \qquad s = 1,2 \end{aligned} \tag{76}$$

5.1. Non-local heat conduction equations

In the present paper, an alternative approach is proposed to combine the three sets of macroscopic equations into a single one and the dependence on the reduced temporal scale can be eliminated.

Defining the mean temperature field as

$$\Phi(x,t) = \langle \phi(x,y,t) \rangle = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \cdots \tag{77}$$

whose time derivative can be expressed as

$$\dot{\Phi} = \Phi_{,t_0} + \varepsilon^2\Phi_{,t_1} \tag{78}$$

$$\ddot{\Phi} = \Phi_{,t_0t_0} + 2\varepsilon^2\Phi_{,t_0t_1} + \varepsilon^4\Phi_{,t_1t_1} \tag{79}$$

Multiplying Eqs. (77)–(79) by ε^0, ε^1 and ε^2, the following relations are obtained

$$\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 = \Phi + \mathrm{O}(\varepsilon^3), \quad \varepsilon(\Phi_0 + \varepsilon\Phi_1) = \varepsilon\Phi + \mathrm{O}(\varepsilon^3), \quad \varepsilon^2\Phi_0 = \varepsilon^2\Phi + \mathrm{O}(\varepsilon^3) \tag{80}$$

$$\Phi_{,t_0} + \varepsilon^2\Phi_{,t_1} = \dot{\Phi}, \quad \varepsilon\Phi_{,t_0} = \varepsilon\dot{\Phi} + \mathrm{O}(\varepsilon^3), \quad \varepsilon^2\Phi_{,t_0} = \varepsilon^2\dot{\Phi} + \mathrm{O}(\varepsilon^3) \tag{81}$$

$$\Phi_{,t_0t_0} + 2\varepsilon^2\Phi_{,t_0t_1} = \ddot{\Phi} + \mathrm{O}(\varepsilon^3), \quad \varepsilon\Phi_{,t_0t_0} = \varepsilon\ddot{\Phi} + \mathrm{O}(\varepsilon^3), \quad \varepsilon^2\Phi_{,t_0t_0} = \varepsilon^2\ddot{\Phi} + \mathrm{O}(\varepsilon^3) \tag{82}$$

The macroscopic equations of heat conduction are expressed as

$$\mathrm{O}(\varepsilon^0):\ \lambda_n(\tau\Phi_{0,t_0t_0} + \Phi_{0,t_0}) - k_n\Phi_{0,xx} = 0 \tag{83}$$

$$\mathrm{O}(\varepsilon^1):\ \lambda_n(\tau\Phi_{1,t_0t_0} + \Phi_{1,t_0}) - k_n\Phi_{1,xx} = 0 \tag{84}$$

$$\mathrm{O}(\varepsilon^2):\ \lambda_n(\tau\Phi_{2,t_0t_0} + \Phi_{2,t_0}) - k_n\Phi_{2,xx} = k_d\Phi_{0,xxxx} - 2\lambda_n\tau\Phi_{0,t_0t_1} - \lambda_n\Phi_{0,t_1} \tag{85}$$

Multiplying Eqs. (83)–(85) by ε^0, ε^1 and ε^2, and then adding them up, we can obtain the macroscopic high-order heat conduction equation for the mean temperature field

$$\begin{aligned}&\lambda_n\tau(\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2)_{,t_0t_0} + \lambda_n(\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2)_{,t_0} - k_n(\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2)_{,xx}\\ &\quad = k_d\varepsilon^2\Phi_{0,xxxx} - 2\lambda_n\tau\varepsilon^2\Phi_{0,t_0t_1} - \lambda_n\varepsilon^2\Phi_{0,t_1}\end{aligned} \tag{86}$$

Substituting Eqs. (80)–(82) into Eq. (86), and neglecting the terms of order higher than ε^2, we have

$$\begin{aligned}&\lambda_n\tau\Phi_{,t_0t_0} + 2\lambda_n\tau\varepsilon^2\Phi_{,t_0t_1} + \lambda_n\Phi_{,t_0} + \lambda_n\varepsilon^2\Phi_{,t_1} - k_n\Phi_{,xx} - k_d\varepsilon^2\Phi_{,xxxx} = 0\\ &\quad \Rightarrow \lambda_n\tau\ddot{\Phi} + \lambda_n\dot{\Phi} - k_n\Phi_{,xx} - k_d\varepsilon^2\Phi_{,xxxx} = 0\end{aligned} \tag{87}$$

In addition, attention is restricted to the approximation and numerical implementation of Eq. (87). The highest spatial derivatives in Eq. (87) is fourth order and therefore C^1 continuity is required for the corresponding finite element implementation. It thus necessitates four boundary conditions to constitute a well-posed boundary value problem. However, for the problem under consideration, there are only two physically meaningful boundary conditions for the mean temperature field. To resolve these difficulties we will attempt to approximate the fourth-order spatial derivative in terms of the mixed second-order spatial-temporal derivative, and to obtain in this way C^0 continuity for the finite element equation.

5.2. Reformulation of heat conduction equations

Multiplying Eq. (83) by ε^2, we have

$$\lambda_n(\tau\varepsilon^2\Phi_{0,t_0t_0} + \varepsilon^2\Phi_{0,t_0}) = k_n\varepsilon^2\Phi_{0,xx} \tag{88}$$

Substituting Eqs. (80)–(82) into Eq. (88), yields

$$\varepsilon^2\Phi_{,xx} = \frac{\lambda_n\varepsilon^2(\tau\ddot{\Phi} + \dot{\Phi})}{k_n} + \mathrm{O}(\varepsilon^3) \tag{89}$$

Substituting Eq. (89) into Eq. (87) and neglecting the terms higher than ε^2, we have

$$\lambda_n(\tau\ddot{\Phi} + \dot{\Phi}) - k_n\Phi_{,xx} - k_m(\tau\ddot{\Phi}_{,xx} + \dot{\Phi}_{,xx}) = 0 \tag{90}$$

where

$$k_m = \frac{\lambda_n k_d \varepsilon^2}{k_n} = \frac{(\alpha(1-\alpha))^2(k_1\lambda_1 - k_2\lambda_2)^2 l^2}{12\lambda_n((1-\alpha)k_1 + \alpha k_2)^2} \tag{91}$$

6. Finite element discretization

The finite element semi-discretization of Eq. (90) will be presented in this section. Since the highest spatial derivatives appearing in Eq. (90) are of second order, the usual C^0 finite element approximation is sufficient. The weak statement of the problem is formulated as follows. $\forall t \in (0, T]$, find $\Phi(x,t) \in S_1(\Omega) \times C^2(0,T]$.

For all admissible test functions $v(x) \in S_0(\Omega)$, where $S_0(\Omega) = \{v(x)|v(x) \in S_1(\Omega) \text{ and } v(x) = 0 \text{ on } \Gamma_\phi\}$, it has

$$\int_\Omega \lambda_n v(\tau\ddot{\Phi} + \dot{\Phi})F\,\mathrm{d}\Omega - \int_\Omega v k_n \Phi_{,xx} F\,\mathrm{d}\Omega - \int_\Omega v k_m(\tau\ddot{\Phi}_{,xx} + \dot{\Phi}_{,xx})F\,\mathrm{d}\Omega = 0 \tag{92}$$

where $S_1(\Omega)$ is the Sobolev space defined as $S_1(\Omega) = \{v(x)|v(x) \in L^2(\Omega), v(x)_{,x} \in L^2(\Omega)\}$, with $L^2(\Omega)$ denoting the set of square-integrable functions over Ω.

Integrating Eq. (92) by parts yields the following weak form equation

$$\begin{aligned}\int_\Omega \lambda_n v(\tau\ddot{\Phi} + \dot{\Phi})F\,\mathrm{d}\Omega + \int_\Omega k_n v_{,x}\Phi_{,x}F\,\mathrm{d}\Omega + \int_\Omega v_{,x} k_m(\tau\ddot{\Phi}_{,x} + \dot{\Phi}_{,x})F\,\mathrm{d}\Omega \\ = k_n\Phi_{,x}v(L)F + k_m(\tau\ddot{\Phi}_{,x} + \dot{\Phi}_{,x})v(L)F\end{aligned} \tag{93}$$

Finite element approximation of the above weak form leads to the semi-discrete equations of heat conduction

$$\mathbf{M}\ddot{\Phi} + \mathbf{C}\dot{\Phi} + \mathbf{K}\Phi = \mathbf{p} \tag{94}$$

where $\Phi(t)$ is the vector of nodal temperature field; $\mathbf{M}$ is the system relaxation consistent matrix; $\mathbf{C}$ and $\mathbf{K}$ are the system heat capacity and heat conduction matrices; $\mathbf{p}$ is the equivalent thermal load vector.

$$\mathbf{M} = \sum_{e=1}^{N_e}\mathbf{m}^e, \quad \mathbf{C} = \sum_{e=1}^{N_e}\mathbf{c}^e, \quad \mathbf{K} = \sum_{e=1}^{N_e}\mathbf{k}^e \tag{95}$$

$$\begin{aligned}\mathbf{m}^e &= \int_{\Omega_e}\lambda_n\tau F\mathbf{N}^\mathrm{T}\mathbf{N}\,\mathrm{d}\Omega + \int_{\Omega_e}k_m\tau F\mathbf{B}^\mathrm{T}\mathbf{B}\,\mathrm{d}\Omega, \\ \mathbf{c}^e &= \int_{\Omega_e}\lambda_n F\mathbf{N}^\mathrm{T}\mathbf{N}\,\mathrm{d}\Omega + \int_{\Omega_e}k_m F\mathbf{B}^\mathrm{T}\mathbf{B}\,\mathrm{d}\Omega, \quad k^e = \int_{\Omega_e}k_n F\mathbf{B}^\mathrm{T}\mathbf{B}\,\mathrm{d}\Omega\end{aligned} \tag{96}$$

$$\mathbf{p} = \left\{Q(t) + \frac{k_m}{k_n}[\tau\ddot{Q}(t) + \dot{Q}(t)]\right\}\Big|_{x=L} \tag{97}$$

where $\mathbf{N}$ is the shape function matrix; $\mathbf{B}$ is the matrix of symmetric gradient of $\mathbf{N}$; $\mathbf{m}^e$ is the element relaxation consistent matrix; $\mathbf{c}^e$ the element matrix of heat capacity matrix; $\mathbf{k}^e$ the element heat conduction matrix.

7. Numerical illustration

We compute in this section the non-Fourier heat conduction problem in a one-dimensional slender bar with the periodic microstructure as shown in Fig. 1. The cross-section of the bar is assumed to be unity. The volume fraction is denoted by f. L is the length of the bar. l is the length of the unit cell. Material properties are denoted as follows: λ is the specific heat of unit volume; k is thermal conductivity; τ is the relaxation time.

In all the cases to be computed, the geometrical parameters are: $L = 20$ μm, $l = 1$ μm. The input heat $Q(t) = Q_0 \cdot t^4 \cdot (t - T)^4$ is supplied at right side of the bar. The adiabatic boundary conditions are imposed along the bar so that no heat transfer occurs between the bar and the ambience. Assuming that the initial temperature $t = 0$ °C is uniform throughout the bar.

7.1. Effects of volume fraction

Material parameters adopted are presented in Table 1. The period of heat load is $T = 6.283 \times 10^{-7}$ s. Fig. 2(a,b) show three curves of midpoint temperature obtained respectively by using (a) finite element model and (b) high-order non-local model for the different cases of volume fraction parameters 0.0, 0.5 and 1.0. The phenomenon of dispersion can be obviously observed in the periodic heterogeneous materials (volume fraction 0.5), but disappears in homogeneous materials (volume fraction 0.0 and 1.0). It can also be observed from this example that the numerical results obtained by the high-order non-local model are good agreement with the FEM model.

7.2. Effects of heat load period

Material parameters used are given in Table 2. In Fig. 3(a,b,c) there are three curves, which indicate respectively the solution obtained by the high-order non-local heat conduction model developed in this paper, the classical homogenization method, and the general (fine) finite element computation. Three kinds of heat load period are computed: (a) $T = 6.283 \times 10^{-7}$ s, (b) $T = 15.708 \times 10^{-7}$ s and (c) $T = 31.416 \times 10^{-7}$ s. From Fig. 3, it can be found that high-order non-local model is always effective, but the classical homogenization model is only valid for the case (c) (see the results in Fig. 3(c)). This is because the classical homogenization model is only effective for the case that the period of heat load is larger than the time required for the heat wave moving through one structural unit cell.

7.3. Effects of heat conduction parameters

This example studies the effects of heat conduction parameters in the numerical computations. Four cases are computed to validate the non-local model and dispersion effects of the numerical model developed. Material parameters used are shown in Tables 3–6. Three curves of temperature are presented in each figure which respectively correspond to high-order non-local model, classical homogenization model and the general (fine) finite element model.

Table 1
Material parameters when $T = 6.283 \times 10^{-7}$ s

	λ (J/m^{3}K)	k (W/mK)	τ (s)
Phase 1	12.0×10^6	50	1.25×10^{-6}
Phase 2	6.0×10^6	25	1.0×10^{-6}

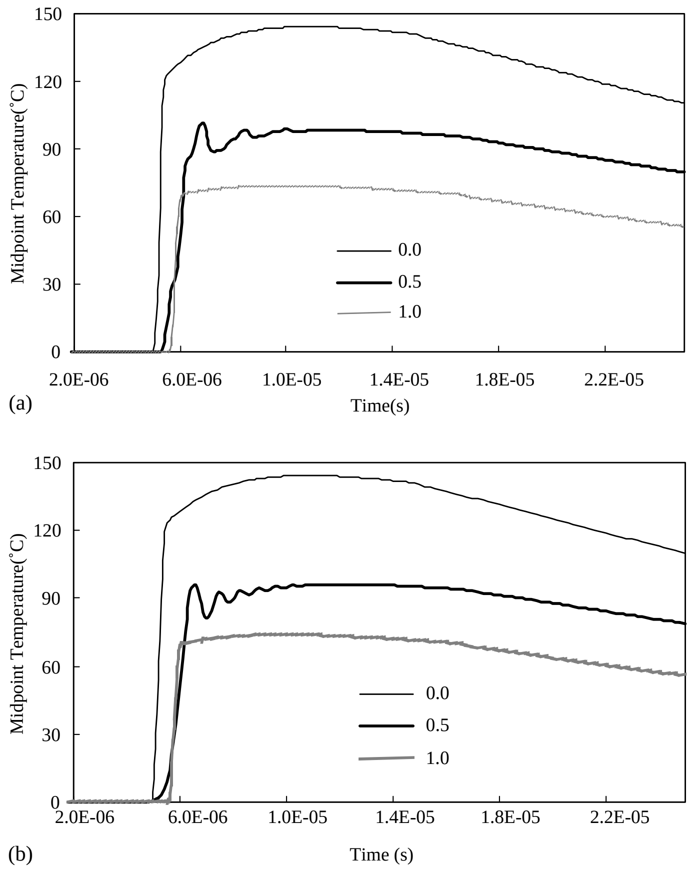

Fig. 2. Temperature response at $x = 10$ μm for volume fractions of 0.0, 0.5 and 1.0. (a) FEM model, (b) high-order non-local model.

Table 2
Material parameters used in Section 7.2

	λ (J/m^{3}K)	k (W/mK)	τ (s)	f
Phase 1	12.0×10^6	50	1.25×10^{-6}	0.5
Phase 2	6.0×10^6	25	1.0×10^{-6}	0.5

The phenomenon of dispersion in non-Fourier heat conduction can be obviously observed in Figs. 4–19. The results obtained by high-order non-local heat conduction model are good agreement with the general finite element solutions, whereas the classical homogenization errors badly for the heterogeneous media.

From the results at $x = 19$ μm, it shows the boundary effect in the homogenization analysis of the heterogeneous materials. The boundary effect is quite obvious and the phenomenon of dispersion is quite strong because the point chosen is not far from the boundary where the heat source is supplied.

From the results at $x = 10$ μm, it can be found that the classical homogenization method has big errors and cannot simulate well the dispersion phenomenon. In contrast to this, the high-order non-local heat conduction model developed has good tendency on the simulation of the diffusion and boundary effects, though it and finite element solutions do not agree identically for all time. Furthermore, despite the shift in phase, the start-up of the temperature response is nearly identical in the high-order non-local and finite

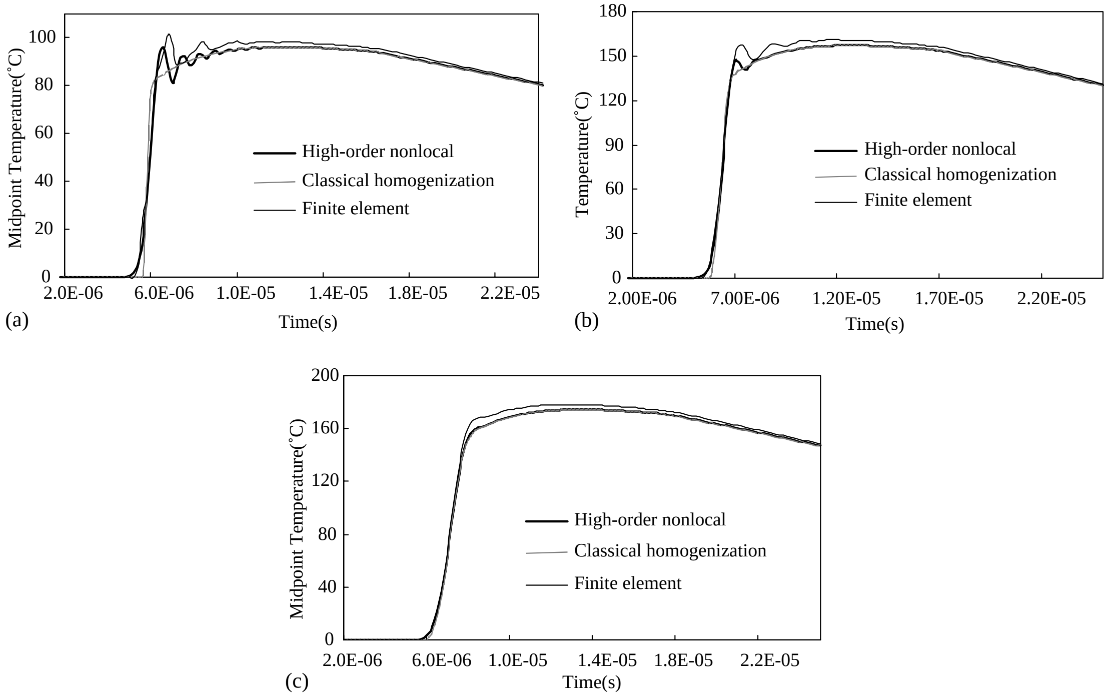

Fig. 3. Temperature response at $x = 10$ μm under various heat loads: (a) $T = 6.283 \times 10^{-7}$ s, (b) $T = 15.708 \times 10^{-7}$ s, (c) $T = 31.416 \times 10^{-7}$ s.

Table 3
Case 1: $\lambda_1 = \lambda_2$, $\tau_1 = \tau_2$, $k_1 \neq k_2$, $T = 4.713 \times 10^{-7}$ s

		λ (J/m^{3}K)	k (W/mK)	τ (s)	f
Material parameters	Phase 1	3.0×10^6	50	1.0×10^{-6}	0.4
(Figs. 4 and 5)	Phase 2	3.0×10^6	10	1.0×10^{-6}	0.6
Material parameters	Phase 1	2.7×10^6	60	2.0×10^{-6}	0.4
(Figs. 6 and 7)	Phase 2	2.7×10^6	15	2.0×10^{-6}	0.6

Table 4
Case 2: $\lambda_1 \neq \lambda_2$, $\tau_1 = \tau_2$, $k_1 \neq k_2$, $T = 6.283 \times 10^{-7}$ s

		λ (J/m^{3}K)	k (W/mK)	τ (s)	f
Material parameters	Phase 1	12.0×10^6	50	1.0×10^{-6}	0.4
(Figs. 8 and 9)	Phase 2	3.0×10^6	10	1.0×10^{-6}	0.6
Material parameters	Phase 1	8.1×10^6	60	2.0×10^{-6}	0.4
(Figs. 10 and 11)	Phase 2	2.7×10^6	15	2.0×10^{-6}	0.6

element models. This implies that the homogenized heat wave speed is correctly modeled through the high-order non-local model.

It is also observed that in Figs. 5, 7, 9 and 11, a better approximation (than Figs. 13, 15, 17 and 19) is provided by high-order non-local heat conduction model in heterogeneous media, because in the cases 1 the

Table 5
5 Case 3: $\lambda_1 = \lambda_2$, $\tau_1 \neq \tau_2$, $k_1 \neq k_2$, $T = 6.283 \times 10^{-7}$ s

		λ (J/m^{3}K)	k (W/mK)	τ (s)	f
Material parameters	Phase 1	3.0×10^6	50	4.0×10^{-6}	0.4
(Figs. 12 and 13)	Phase 2	3.0×10^6	10	1.0×10^{-6}	0.6
Material parameters	Phase 1	2.7×10^6	60	6.0×10^{-6}	0.4
(Figs. 14 and 15)	Phase 2	2.7×10^6	15	2.0×10^{-6}	0.6

Table 6
Case 4: $\rho_1 c_1 \neq \rho_2 c_2$, $\tau_1 \neq \tau_2$, $k_1 \neq k_2$, $T = 7.854 \times 10^{-7}$ s

		λ (J/m^{3}K)	k (W/mK)	τ (s)	f
Material parameters	Phase 1	12.0×10^6	50	4.0×10^{-6}	0.4
(Figs. 16 and 17)	Phase 2	3.0×10^6	10	1.0×10^{-6}	0.6
Material parameters	Phase 1	8.1×10^6	60	6.0×10^{-6}	0.4
(Figs. 18 and 19)	Phase 2	2.7×10^6	15	2.0×10^{-6}	0.6

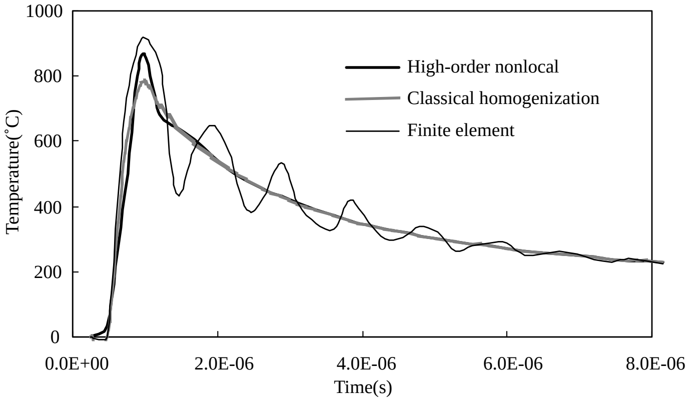

Fig. 4. Case 1: temperature response at $x = 19$ μm.

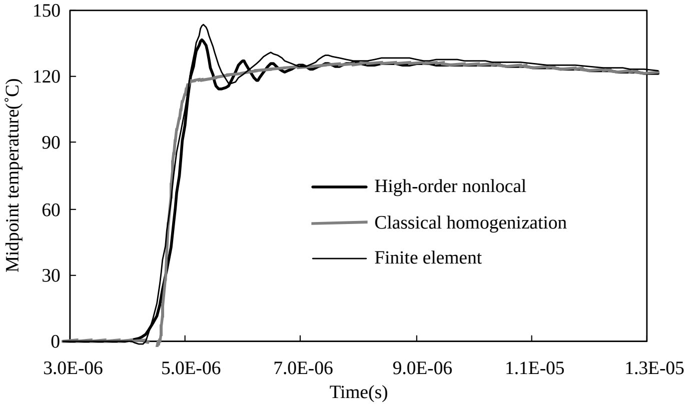

Fig. 5. Case 1: temperature response at $x = 10$ μm.

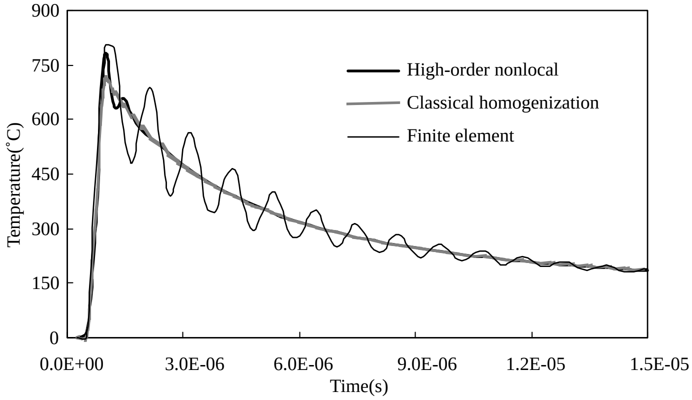

Fig. 6. Case 1: temperature response at $x = 19$ μm.

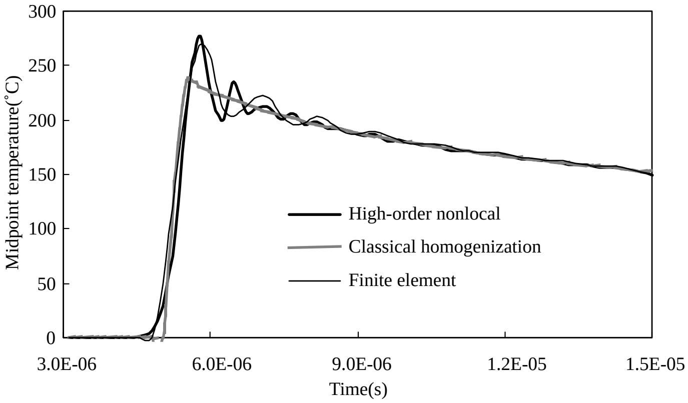

Fig. 7. Case 1: temperature response at $x = 10$ μm.

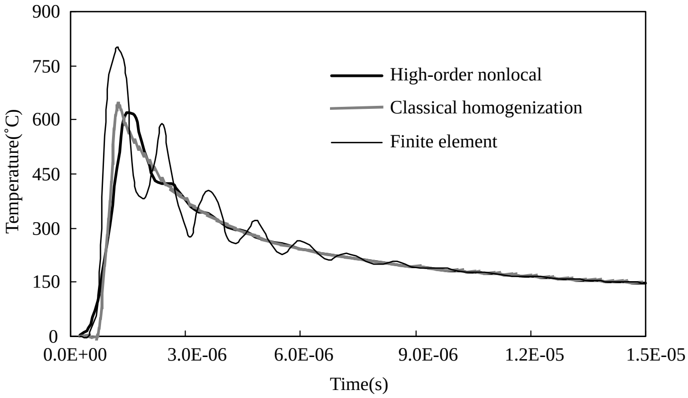

Fig. 8. Case 2: temperature response at $x = 19$ μm.

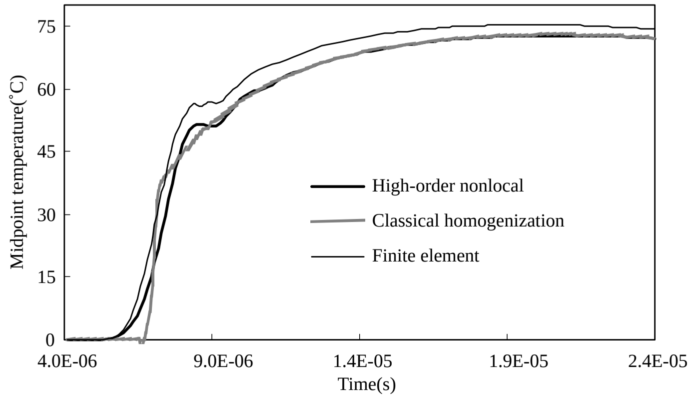

Fig. 9. Case 2: temperature response at $x = 10$ μm.

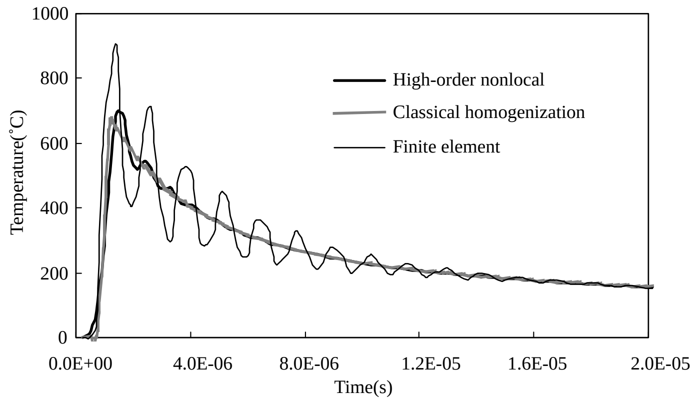

Fig. 10. Case 2: temperature response at $x = 19$ μm.

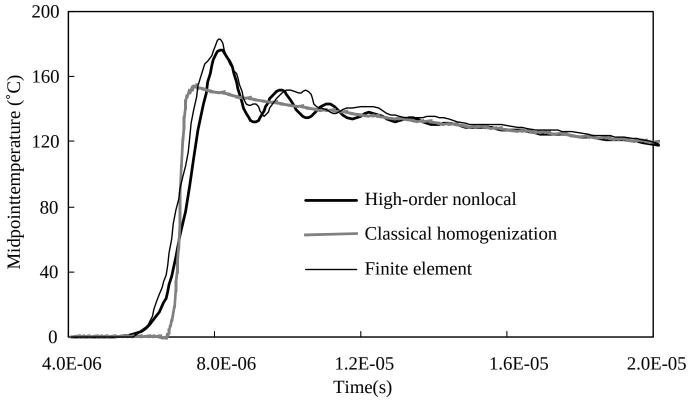

Fig. 11. Case 2: temperature response at $x = 10$ μm.

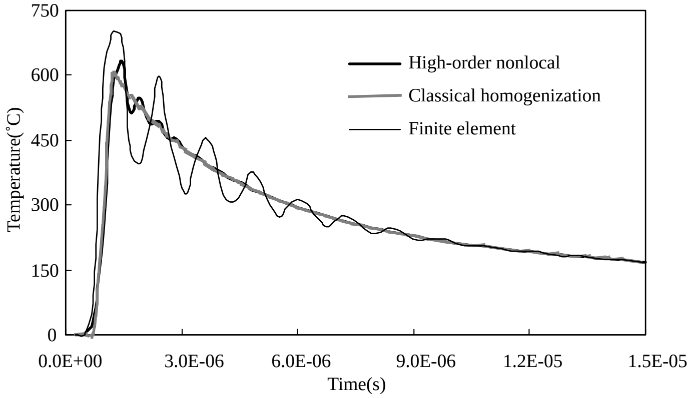

Fig. 12. Case 3: temperature response at $x = 19$ μm.

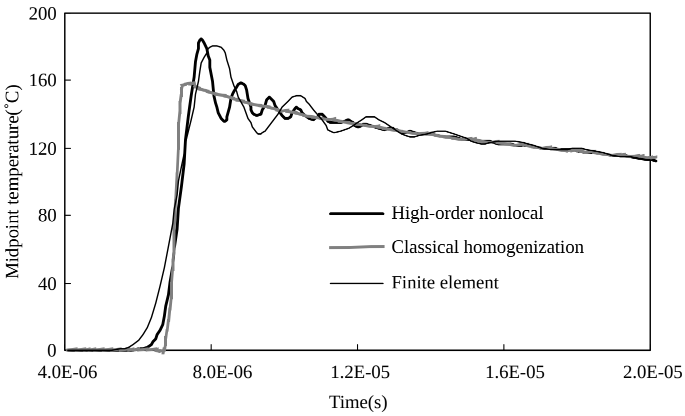

Fig. 13. Case 3: temperature response at $x = 10$ μm.

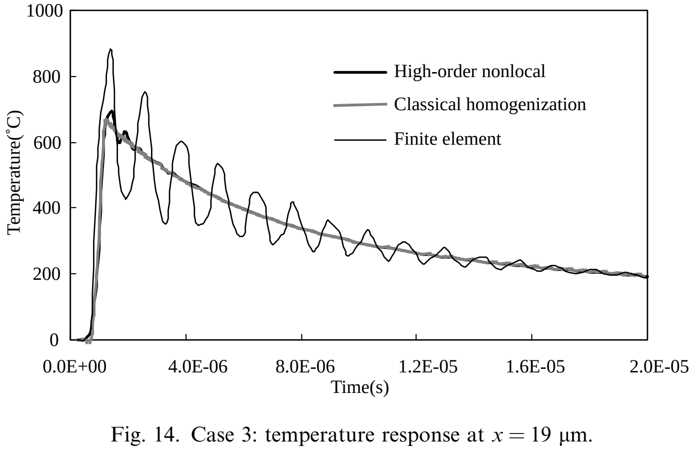

Fig. 14. Case 3: temperature response at $x = 19$ μm.

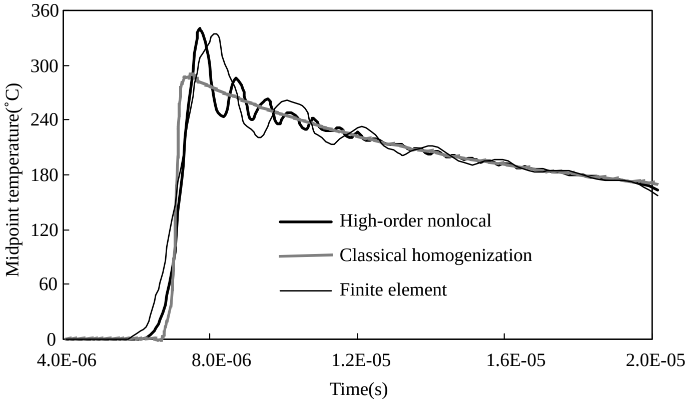

Fig. 15. Case 3: temperature response at $x = 10$ μm.

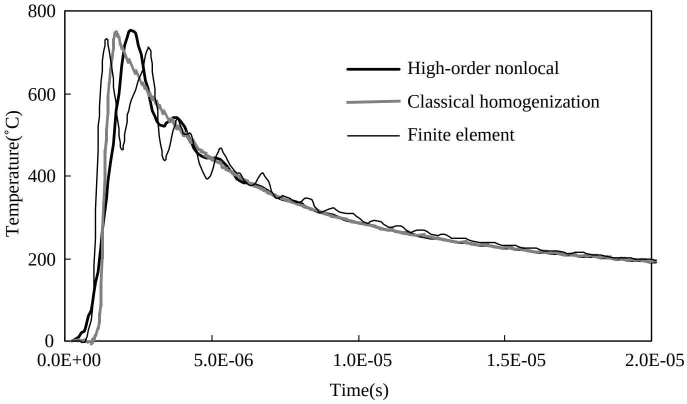

Fig. 16. Case 4: temperature response at $x = 19$ μm.

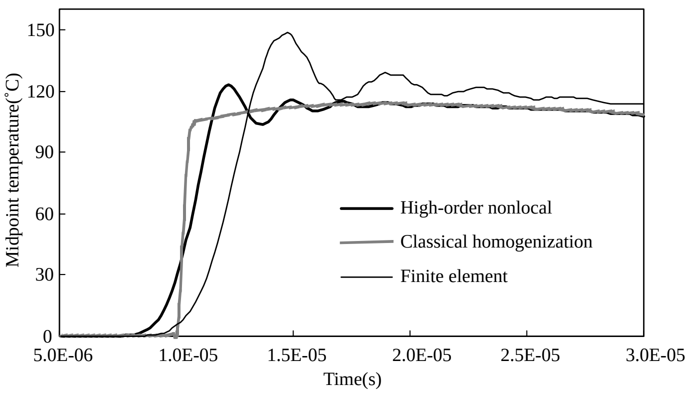

Fig. 17. Case 4: temperature response at $x = 10$ μm.

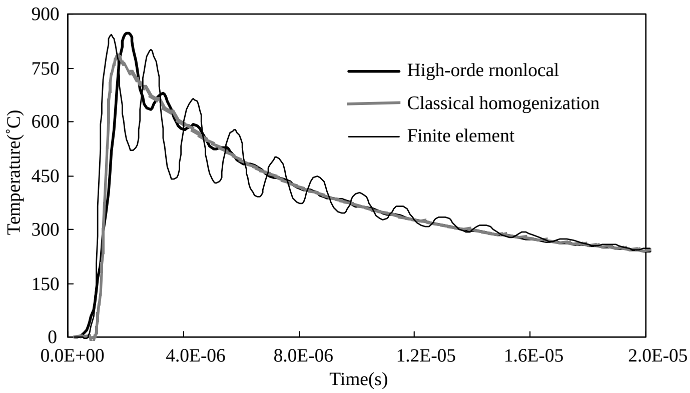

Fig. 18. Case 4: temperature response at $x = 19$ μm.

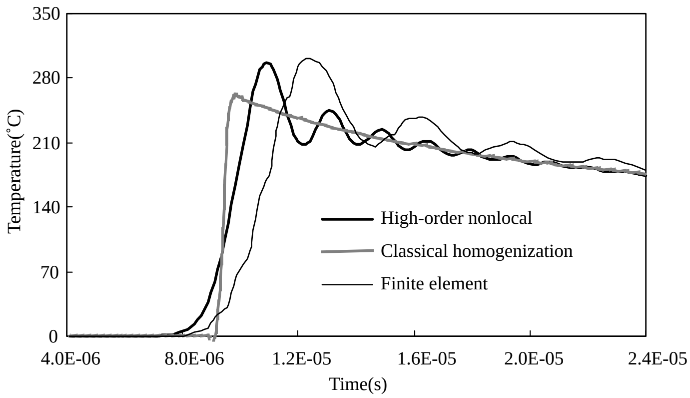

Fig. 19. Case 4: temperature response at $x = 10$ μm.

conditions of $\lambda_1 = \lambda_2$, $\tau_1 = \tau_2$ hold, the high-order non-local heat conduction model is derived without any other assumptions. While in the cases 4, due to that $\lambda_1 \neq \lambda_2$, $\tau_1 \neq \tau_2$, it is necessary to introduce an assumption to average these parameters in the model derivative procedure, which will induce naturally some errors in the numerical computations. This can be demonstrated by the results shown in Figs. 13, 15, 17 and 19. Even though, the non-local model can still show the advantages on the simulation of the dispersion and boundary effect in the heterogeneous materials.

8. Conclusions

The non-classical heat conduction problem under extreme conditions is one of the hotspot problems in current research field of heat conduction and has strong potential for engineering applications. This paper represents the multiple scale numerical simulation procedure for the solution of heat conduction problems under high frequency impulse heat load in macroscopically isotropic heterogeneous media. Two different kinds of scales, the amplified spatial scale and the reduced temporal scale are introduced respectively to describe the fluctuation effect with multiple temporal scales in local heterogeneous media. Homogenized constitutive equations and different orders of boundary value problems are obtained by utilizing the asymptotic analysis method with multiple spatial and temporal scales. Homogenized constitutive equations and different orders of governing equations of boundary value problems are derived by utilizing the asymptotic analysis method with multiple spatial and temporal scales. Numerical examples demonstrate the validity of the non-local model proposed. It can be concluded that classical homogenization theory has certain limitation for solving of the problems under special load case, particularly the impulse load with high frequency, and thus high order homogenization theory is required for better modeling of the corresponding problems.

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